Discrete Hashimoto surfaces and a doubly discrete smoke ring flow

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Abstract

Bäcklund transformations for smooth and "space discrete" Hashimoto surfaces are discussed and a geometric interpretation is given. It is shown that the complex curvature of a discrete space curve evolves with the discrete nonlinear Schrödinger equation (NLSE) of Ablowitz and Ladik, when the curve evolves with the Hashimoto or smoke ring flow. A doubly discrete Hashimoto flow is derived and it is shown, that in this case the complex curvature of the discrete curve obeys Ablovitz and Ladik's doubly discrete NLSE. Elastic curves (curves that evolve by rigid motion only under the Hashimoto flow) in the discrete and doubly discrete case are shown to be the same.

There is an online version of this paper, that can be viewed using any recent web browser that has JAVA support enabled. It includes two additional java applets. It can be found at http://www-sfb288.math.tu-berlin.de/Publications/online/smokeringsOnline/index.html

1 Introduction

Many of the surfaces that can be described by integrable equations have been discretized. Among them are surfaces of constant negative Gaussian curvature, surfaces of constant mean curvature, minimal surfaces, and affine spheres. This paper continues the program by adding Hashimoto surfaces to the list. These surfaces are obtained by evolving a regular space curve γ by the Hashimoto or *smoke ring flow*

$$\dot{\gamma} = \gamma' \times \gamma''.$$

As shown by Hashimoto [6] this evolution is directly linked to the famous nonlinear Schrödinger equation (NLSE)

$$i\dot{\Psi} + \Psi'' + \frac{1}{2}|\Psi|^2\Psi = 0.$$

In [1] and [2] Ablowitz and Ladik gave a differential-difference and a difference-difference discretization of the NLSE. In [7] the author shows[1] that they correspond to a Hashimoto flow on discrete curves (i. e. polygons) [3, 4] and a doubly discrete Hashimoto flow respectively. This discrete evolution is derived in section 3.2.2 from a discretization of the Bäcklund transformations for regular space curves and Hashimoto surfaces.

In Section 2 a short review of the smooth Hashimoto flow and its connection to the isotropic Heisenberg magnet model and the nonlinear Schrödinger equation is given. It is shown that the solutions to the auxiliary problems of these integrable equations serve as frames for the Hashimoto surfaces and a Sym formula is derived. In section 2.2.1 the dressing procedure or Bäcklund transformation is discussed and applied on the vacuum. A geometric interpretation of this transformation as a generalization of the Traktrix construction for a curve is given.

In Section 3 the same program is carried out for the Hashimoto flow on discrete curves. Then in Section 4 special double Bäcklund transformations (for discrete curves) are singled out to get a unique evolution which serves as our doubly discrete Hashimoto flow.

Elastic curves (curves that evolve by rigid motion under the Hashimoto flow) are discussed in all these cases. It turns out that discrete elastic curves for the discrete and the doubly discrete Hashimoto flow coincide.

Through this paper we use a quaternionic description. Quaternions are the algebra generated by 1, $\mathfrak{i}$, $\mathfrak{j}$, and $\mathfrak{k}$ with the relations $\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = -1$, $\mathfrak{i}\mathfrak{j} = K, \mathfrak{j}\mathfrak{k} = \mathfrak{i}$, and $\mathfrak{k}\mathfrak{i} = \mathfrak{j}$. Real and imaginary part of a quaternion are defined in an obvious manner: If $q = \alpha + \beta\mathfrak{i} + \gamma\mathfrak{j} + \delta\mathfrak{k}$ we set $\mathrm{Re}(q) = \alpha$ and $\mathrm{Im}(q) = \beta\mathfrak{i} + \gamma\mathfrak{j} + \delta\mathfrak{k}$. Note that unlike in the complex case the imaginary part is not a real number. We identify the 3-dimensional euclidian space with the imaginary quaternions i. e. the span of $\mathfrak{i}$, $\mathfrak{j}$, and $\mathfrak{k}$. Then for two imaginary quaternions q, r the following formula holds:

$$qr = -\langle q, r\rangle + q \times r$$

with $\langle \cdot, \cdot \rangle$ and $\cdot \times \cdot$ denoting the usual scalar and cross products of vectors in 3-space. A rotation of an imaginary quaternion around the axis $r, |r| = 1$ with angle ϕ can be written as conjugation with the unit length quaternion $(\cos\frac{\phi}{2} + \sin\frac{\phi}{2} r)$.

Especially when dealing with the Lax representations of the various equations it will be convenient to identify the quaternions with complex 2 by 2 matrices:

$$\mathfrak{i} = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \mathfrak{j} = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\mathfrak{k} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

[1] The equivalence for the differential-difference case appeared first in [8].

2 The Hashimoto flow, the Heisenberg flow and the nonlinear Schrödinger equation

Let $\gamma : \mathbb{R} \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ be an arclength parametrized regular curve and $\mathcal{F} : \mathbb{R} \to \mathbb{H}^*$ be a parallel frame for it, i. e.

$$\begin{aligned} \mathcal{F}^{-1}\mathfrak{i}\mathcal{F} &= \gamma' = \gamma_x &(1)\\ (\mathcal{F}^{-1}\mathfrak{j}\mathcal{F})' &\parallel \gamma'. &(2) \end{aligned}$$

The second equation says that $\mathcal{F}^{-1}\mathfrak{j}\mathcal{F}$ is a parallel section in the normal bundle of γ. which justifies the name. Moreover let $A = \mathcal{F}'\mathcal{F}^{-1}$ be the logarithmic derivative of $\mathcal{F}$. Equation (2) gives, that A must lie in the $\mathfrak{j}$-$\mathfrak{k}$-plane and thus can be written as

$$A = -\frac{\Psi}{2}\mathfrak{k} \tag{3}$$

with $\Psi \in \operatorname{span}(1, \mathfrak{i}) \cong \mathbb{C}$.

Definition 1 *We call Ψ the complex curvature of γ.*

Now let us evolve γ with the following flow:

$$\dot{\gamma} = \gamma' \times \gamma'' = \gamma'\gamma''. \tag{4}$$

Here $\dot{\gamma}$ denotes the derivative in time. This is an evolution in binormal direction with velocity equal to the (real) curvature. It is known as the *Hashimoto* or *smoke ring flow*. Hashimoto was the first to show, that under this flow the complex curvature Ψ of γ solves the nonlinear Schrödinger equation (NLSE) [6]

$$\mathfrak{i}\dot{\Psi} + \Psi'' + \frac{1}{2}|\Psi|^2\Psi = 0. \tag{5}$$

or written for A:

$$\mathfrak{i}\dot{A} + A'' = 2A^3. \tag{6}$$

Definition 2 *The surfaces $\gamma(x, t)$ wiped out by the flow given in equation (4) are called Hashimoto surfaces.*

Equation (5) arises as the zero curvature condition $\widehat{L}_t - \widehat{M}_x + [\widehat{L}, \widehat{M}] = 0$ of the system

$$\begin{aligned} \widehat{\mathcal{F}}_x(\mu) &= \widehat{L}(\mu)\widehat{\mathcal{F}}(\mu)\\ \widehat{\mathcal{F}}_t(\mu) &= \widehat{M}(\mu)\widehat{\mathcal{F}}(\mu) \end{aligned} \tag{7}$$

with

$$\begin{aligned} \widehat{L}(\mu) &= \mu\mathfrak{i} - \tfrac{\Psi}{2}\mathfrak{k}\\ \widehat{M}(\mu) &= \tfrac{|\Psi|^2}{4}\mathfrak{i} + \tfrac{\Psi_x}{2}\mathfrak{j} - 2\mu\widehat{L}(\mu). \end{aligned} \tag{8}$$

To make the connection to the description with the parallel frame $\mathcal{F}$ we add torsion to the curve γ by setting

$$A(\mu) = e^{-2\mu x \mathfrak{i}} \Psi \mathfrak{k}.$$

This gives rise to a family of curves $\gamma(\mu)$ the so-called *associated family* of γ. Now one can gauge the corresponding parallel frame $\mathcal{F}(\mu)$ with $e^{\mu x \mathfrak{i}}$ and get

$$(e^{\mu x \mathfrak{i}} \mathcal{F}(\mu))_x = ((e^{\mu x \mathfrak{i}})_x e^{-\mu x \mathfrak{i}} + e^{\mu x \mathfrak{i}} A(\mu) e^{-\mu x \mathfrak{i}}) e^{\mu x \mathfrak{i}} \mathcal{F}(\mu) = L(\mu) e^{\mu x \mathfrak{i}} \mathcal{F}(\mu)$$

with $L(\mu)$ as in (7). So above $\widehat{\mathcal{F}}(\mu) = e^{\mu x \mathfrak{i}} \mathcal{F}(\mu)$ is for each t_0 a frame for the curve $\gamma(x, t_0)$.

Theorem 1 (Sym formula) *Let $\Psi(x,t)$ be a solution of the NLSE (equation (5)). Then up to an euclidian motion the corresponding Hashimoto surface $\gamma(x,t)$ can be obtained by*

$$\gamma(x,t) = \widehat{\mathcal{F}}^{-1} \widehat{\mathcal{F}}_\lambda |_{\lambda=0} \tag{9}$$

where $\widehat{\mathcal{F}}$ is a solution to (7).

Proof Obviously $\widehat{\mathcal{F}}|_{\lambda=0}(x,t_0)$ is a parallel frame for each $\gamma(x,t_0)$. So writing $\widehat{\mathcal{F}}(x,t_0)|_{\lambda=0} =: \mathcal{F}(x)$, one easily computes $(\widehat{\mathcal{F}}^{-1} \widehat{\mathcal{F}}_\lambda|_{\lambda=0})_x = \mathcal{F}^{-1} \mathfrak{i} \mathcal{F} = \gamma_x$ and $(\widehat{\mathcal{F}}^{-1} \widehat{\mathcal{F}}_\lambda|_{\lambda=0})_y = \mathcal{F}^{-1} \Psi \mathfrak{k} \mathcal{F}$. But $\gamma_t = \gamma_x \gamma_{xx} = \mathcal{F}^{-1} \Psi \mathfrak{k} \mathcal{F}$. $\square$

If one differentiates equation (4) with respect to x one gets the so-called isotropic Heisenberg magnet model (IHM):

$$\dot{S} = S \times S'' = S \times S_{xx} \tag{10}$$

with $S = \gamma'$. This equation arises as zero curvature condition $U_t - V_x + [U,V] = 0$ with matrices

$$\begin{array}{rcl} U(\lambda) & = & \lambda S \\ V(\lambda) & = & -2\lambda^2 S - \lambda S' S \end{array} \tag{11}$$

In fact if G is a solution to

$$\begin{array}{c} G_x = U(\lambda) G \\ G_t = V(\lambda) G \end{array} \tag{12}$$

it can be viewed as a frame for the Hashimoto surface too and one has a similar Sym formula:

$$\gamma(x,t) = G^{-1} G_\lambda |_{\lambda=0} \tag{13}$$

The system (12) is known to be gauge equivalent to (7) [5].

2.1 Elastic curves

The stationary solutions of the NLSE (i. e. the curves that evolve by rigid motion under the Hashimoto flow) are known to be the *elastic curves* [3]. They are the critical points of the functional

$$E(\gamma) = \int \kappa^2$$

with $\kappa = |\Psi|$ the curvature of γ. The fact that they evolve by rigid motion under the Hashimoto flow can be used to give a characterization by their complex curvature Ψ only: When the curve evolves by rigid motion Ψ may get a phase factor only. Thus $\dot{\Psi} = c i\Psi$. Inserted into equation (5) this gives

$$\Psi'' = (c - \frac{1}{2}|\Psi|^2)\Psi. \tag{14}$$

2.2 Bäcklund transformations for smooth space curves and Hashimoto surfaces

Now we want to describe the dressing procedure or Bäcklund transformation for the IHM model and the Hashimoto surfaces. This is a method to generate new solutions of our equations from a given one in a purely algebraic way. Afterwards we give some geometric interpretation for this transformation.

2.2.1 Algebraic description of the Bäcklund transformation

Theorem 2 *Let G be a solution to equations (12) with U and V as in (11) (i. e. $U(1)$ solves the IHM model). Choose $\lambda_0, s_0 \in \mathbb{C}$. Then $\widetilde{G}(\lambda) := B(\lambda)G(\lambda)$ with $B(\lambda) = (\mathbb{I} + \lambda\rho), \rho \in \mathbb{H}$ defined by the conditions that $\lambda_0, \overline{\lambda}_0$ are the zeroes of* $\det(B(\lambda))$ *and*

$$\widetilde{G}(\lambda_0)\begin{pmatrix} s_0 \\ 1 \end{pmatrix} = 0 \quad \text{and} \quad \widetilde{G}(\overline{\lambda}_0)\begin{pmatrix} 1 \\ -\overline{s}_0 \end{pmatrix} = 0 \tag{15}$$

solves a system of the same type. In particular $\widetilde{U}(1) = \widetilde{G}_x(1)\widetilde{G}^{-1}(1)$ solves again the Heisenberg magnet model (10).

Proof We define $\widetilde{U}(\lambda) = \widetilde{G}_x\widetilde{G}^{-1}$ and $\widetilde{V}(\lambda) = \widetilde{G}_t\widetilde{G}^{-1}$. Equation (15) ensures that $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ are smooth at λ_0 and $\overline{\lambda}_0$. Using $\widetilde{U}(\lambda) = B_x(\lambda)B^{-1}(\lambda) + B(\lambda)U(\lambda)B^{-1}(\lambda)$ this in turn implies that $\widetilde{U}(\lambda)$ has the form $\widetilde{U}(\lambda) = \lambda\widetilde{S}$ for some $\widetilde{S}$.

Since the zeroes of $\det(B(\lambda))$ are fixed we know that $r := \mathrm{Re}(\rho)$ and $l := |\,\mathrm{Im}(\rho)|$ are constant. We write $\rho = r + v$.

One gets $\widetilde{S} = S + v_x$ and

$$v_x = \frac{2rl}{r^2+l^2}\frac{v \times S}{l} + \frac{2l^2}{r^2+l^2}\frac{\langle v, S\rangle}{l^2}v - \frac{2l^2}{r^2+l^2}S. \tag{16}$$

This can be used to show $|\widetilde{S}| = 1$.

Again equation (15) ensures that $\widetilde{V}(\lambda) = \lambda^2 X + \lambda Y$ for some X and Y. But then the integrability condition $\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}]$ gives up to a factor c and possible constant real parts x and y that X and Y are fixed to be $X = x + c\widetilde{S}_x\widetilde{S} + d\widetilde{S}$ and $Y = y + 2c\widetilde{S}$. The additional term $d\widetilde{S}$ in X corresponds to the (trivial) tangential flow which always can be added. The form $\widetilde{V}(\lambda) = B_t(\lambda)B^{-1}(\lambda) + B(\lambda)V(\lambda)B^{-1}(\lambda)$ gives $c = -1$ and $d = 0$. Thus one ends up with $\widetilde{V}(\lambda) = -2\lambda^2\widetilde{S} - \lambda\widetilde{S}_x\widetilde{S}$. □

So we get a four parameter family (λ_0 and s_0 give two real parameters each) of transformations for our curve γ that are compatible with the Hashimoto flow. They correspond to the four parameter family of Bäcklund transformations of the NLSE.

Example Let us do this procedure in the easiest case: We choose $S \equiv \mathfrak{i}$ (or $\gamma(x,t) = x\mathfrak{i}$) which gives

$$G(\lambda) = \exp((\lambda x - 2\lambda^2 t)\mathfrak{i}) = \begin{pmatrix} e^{i(\lambda x - 2\lambda^2 t)} & 0 \\ 0 & e^{-i(\lambda x - 2\lambda^2 t)} \end{pmatrix}.$$

After choosing λ_0 and s_0 and writing $\rho = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ one gets with equation (15)

$$\begin{aligned} -e^{i(\lambda_0 x - 2\lambda_0^2 t)} &= \lambda_0(e^{i(\lambda_0 x - 2\lambda_0^2 t)}a + s_0 e^{-i(\lambda_0 x - 2\lambda_0^2 t)}b \\ s_0 e^{-i(\lambda_0 x - 2\lambda_0^2 t)} &= \lambda_0(e^{i(\lambda_0 x - 2\lambda_0^2 t)}\overline{b} - s_0 e^{-i(\lambda_0 x - 2\lambda_0^2 t)}\overline{a}). \end{aligned} \tag{17}$$

These equations can be solved for a and b :

$$\begin{aligned} a &= -\frac{\frac{1}{\lambda_0} + \frac{s_0\overline{s}_0}{\overline{\lambda}_0}e^{-2i(\lambda_0 - \overline{\lambda}_0)x + 4i(\lambda_0^2 - \overline{\lambda}_0^2)t}}{1 + s_0\overline{s}_0 e^{-2i(\lambda_0 - \overline{\lambda}_0)x + 4i(\lambda_0^2 - \overline{\lambda}_0^2)t}} \\ b &= \overline{s}_0 e^{2i\overline{\lambda}_0 x - 4i\overline{\lambda}_0^2 t}\frac{\frac{1}{\overline{\lambda}_0} - \frac{1}{\lambda_0}}{1 + s_0\overline{s}_0 e^{-2i(\lambda_0 - \overline{\lambda}_0)x + 4i(\lambda_0^2 - \overline{\lambda}_0^2)t}} \end{aligned} \tag{18}$$

Using the Sym formula (13) one can immediately write the formula for the resulting Hashimoto surface $\widetilde{\gamma}$:

$$\widetilde{\gamma} = \mathrm{Im}(\rho) + \gamma = \begin{pmatrix} \mathrm{Im}(a) + ix & b \\ -\overline{b} & -\mathrm{Im}(a) - ix \end{pmatrix}.$$

The need for taking the imaginary part is due to the fact that we did not normalize $B(\lambda)$ to $\det(B(\lambda)) = 1$.

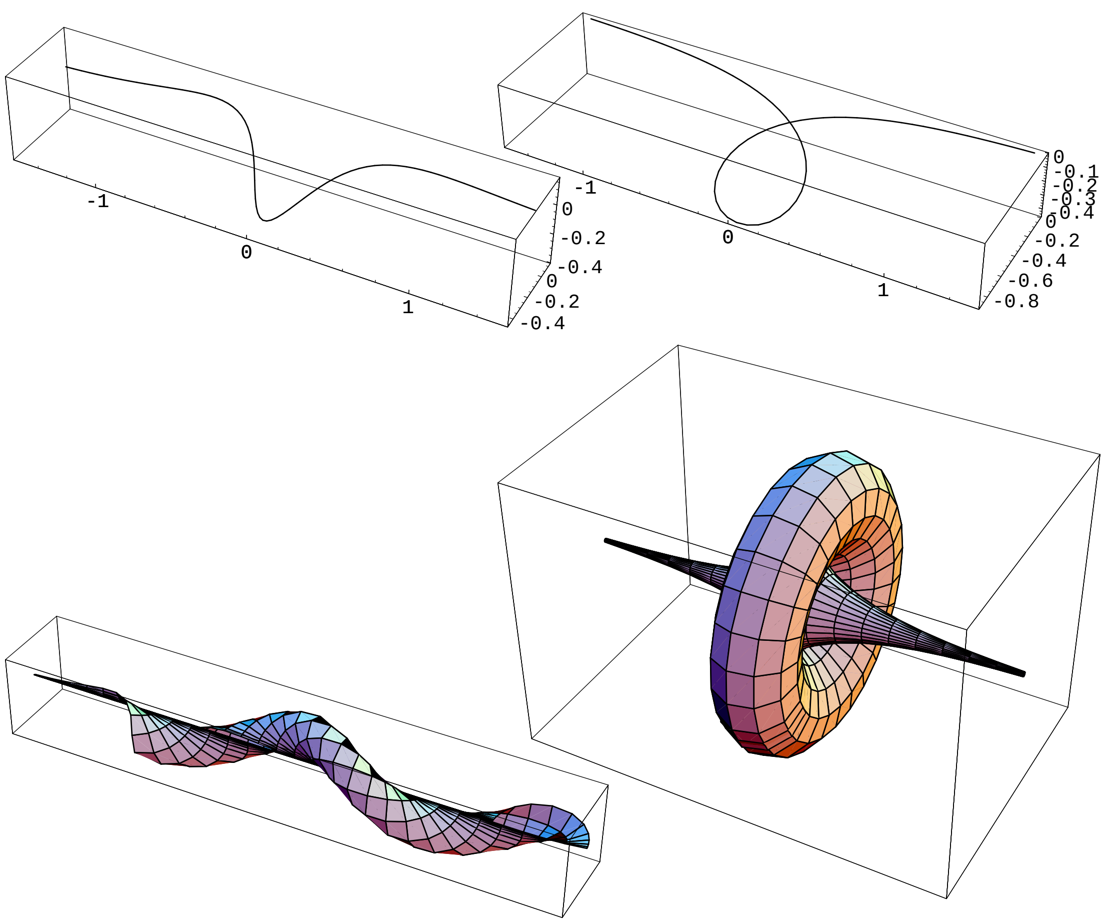

Figure 1: Two dressed straight lines and the corresponding Hashimoto surfaces

If one wants to have the result in a plane $\arg b$ should be constant. This can be achieved by choosing $\lambda \in i\mathbb{R}$. Figure 1 shows the result for $s_0 = 0.5 + i$ and $\lambda_0 = 1 - i$ and $\lambda_0 = -i$ respectively.

Of course one can iterate the dressing procedure to get new curves (or surfaces) and it is a natural question how many one can get. This leads immediately to the Bianchi permutability theorem

Theorem 3 (Bianchi permutability) *Let $\widetilde{\gamma}$ and $\widehat{\gamma}$ be two Bäcklund transforms of γ. Then there is a unique Hashimoto surface $\widehat{\widetilde{\gamma}}$ that is Bäcklund transform of $\widetilde{\gamma}$ and $\widehat{\gamma}$.*

Proof Let $G, \widehat{G}$, and $\widetilde{G}$ be the solutions to (12) corresponding to $\gamma, \widehat{\gamma}$, and $\widetilde{\gamma}$. One has $\widehat{G} = \widehat{B}G$ and $\widetilde{G} = \widetilde{B}G$ with $\widehat{B} = \mathbb{I} + \lambda\widehat{\rho}$ and $\widetilde{B} = \mathbb{I} + \lambda\widetilde{\rho}$. The ansatz $\widetilde{\widehat{B}}\widehat{G} = \widehat{\widetilde{B}}\widetilde{G}$ leads to the compatability condition $\widetilde{\widehat{B}}\widehat{B} = \widehat{\widetilde{B}}\widetilde{B}$ or

$$(\mathbb{I} + \lambda\widetilde{\widehat{\rho}})(\mathbb{I} + \lambda\widehat{\rho}) = (\mathbb{I} + \lambda\widehat{\widetilde{\rho}})(\mathbb{I} + \lambda\widetilde{\rho}) \tag{19}$$

which gives:

$$\begin{aligned} \widetilde{\widehat{\rho}} &= (\widehat{\rho} - \widetilde{\rho})\,\widetilde{\rho}\,(\widehat{\rho} - \widetilde{\rho})^{-1} \\ \widehat{\widetilde{\rho}} &= (\widehat{\rho} - \widetilde{\rho})\,\widehat{\rho}\,(\widehat{\rho} - \widetilde{\rho})^{-1}. \end{aligned} \tag{20}$$

Thus $\widetilde{\widehat{B}}$ and $\widehat{\widetilde{B}}$ are completely determined. To show that they give dressed solutions we note that since $\det \widetilde{\widehat{B}} \det \widehat{B} = \det \widehat{\widetilde{B}} \det \widetilde{B}$ the zeroes of $\det \widetilde{\widehat{B}}$ are the same as the ones of $\det \widetilde{B}$ (and the ones of $\det \widehat{\widetilde{B}}$ coincide with those of $\det \widehat{B}$). Therefore they do not depend on x and t. Moreover at these points the kernel of $\widetilde{\widehat{B}}\widehat{G}$ coincides with the one of $\widetilde{G}$. Thus it does not depend on x or t either. Now theorem 2 gives the desired result. $\square$

2.2.2 Geometry of the Bäcklund transformation

As before let $\gamma : I \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ be an arclength parametrized regular curve. Moreover let $v : I \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$, $|v| = l$ be a solution to the following system:

$$\begin{aligned} \widehat{\gamma} &= \gamma + \tfrac{1}{2}v \\ \widehat{\gamma}' &\parallel v. \end{aligned} \tag{21}$$

Then $\widehat{\gamma}$ is called a Traktrix of γ. The forthcoming definition in this section is motivated by the following observation: If we set $\widetilde{\gamma} = \gamma + v$ it is again an arclength parametrized curve and $\widehat{\gamma}$ is a Traktrix of $\widetilde{\gamma}$ too. One can generalize this in the following way:

Lemma 4 *Let $v : I \to \operatorname{Im}\mathbb{H}$ be a vector field along γ of constant length l satisfying*

$$v' = 2\sqrt{b - b^2}\,\frac{v \times \gamma'}{l} + 2b\frac{< v, \gamma' >}{l^2}v - 2b\gamma' \tag{22}$$

with $0 \leq b \leq 1$. Then $\widetilde{\gamma} = \gamma + v$ is arclength parametrized.

Proof Obviously the above transformation coincides with the dressing described in the last section with $b = \frac{l^2}{r^2+l^2}$ in formula (16). This proves the lemma. $\square$

So $\mathrm{Im}(\rho)$ from theorem 2 is nothing but the difference vector between the original curve and the Bäcklund transform. Note that in the case $b = 1$ one gets the above Traktrix construction, that is for $\widehat{\gamma} = \gamma + v$ holds $\widehat{\gamma}' \parallel v$. This motivates the following

Definition 3 *The curve* $\widehat{\gamma} = \gamma + \frac{1}{2}v$ *with* v *as in lemma 4 is called a* twisted Traktrix *of the curve* γ *and* $\widetilde{\gamma} = \gamma + v$ *is called a* Bäcklund transform *of* γ*.*

Moreover equation (22) gives that $v' \perp v$ and therefore $|v| \equiv const$. Since $v = \widetilde{\gamma} - \gamma$ we see that the Bäcklund transform is in constant distance to the original curve.

3 The Hashimoto flow, the Heisenberg flow, and the nonlinear Schrödinger equation in the discrete case

In this section we give a short review on the discretization (in space) of the Hashimoto flow, the isotropic Heisenberg magnetic model, and the nonlinear Schrödinger equation. For more details on this topic see [5, 3, 4] and [7].

We call a map $\gamma : \mathbb{Z} \to \mathrm{Im}\,\mathbb{H}$ a discrete regular curve if any two successive points do not coincide. It will be called arclength parametrized curve, if $|\gamma_{n+1} - \gamma_n| = 1$ for all $n \in \mathbb{Z}$. We will use the notation $S_n := \gamma_{n+1} - \gamma_n$. The binormals of the discrete curve can be defined as $\frac{S_n \times S_{n-1}}{|S_n \times S_{n-1}|}$.

There is a natural discrete analog of a parallel frame:

Definition 4 *A* discrete parallel frame *is a map* $\mathcal{F} : \mathbb{Z} \to \mathbb{H}^*$ *with* $|\mathcal{F}_k| = 1$ *satisfying*

$$\begin{aligned} S_n &= \mathcal{F}_n^{-1}\mathfrak{i}\mathcal{F}_n & (23)\\ \mathrm{Im}\left((\mathcal{F}_{n+1}^{-1}\mathfrak{j}\mathcal{F}_{n+1})(\mathcal{F}_n^{-1}\mathfrak{j}\mathcal{F}_n)\right) &\parallel \mathrm{Im}\left(S_{n+1}S_n\right). & (24)\end{aligned}$$

Again we set $\mathcal{F}_{n+1} = A_n\mathcal{F}_n$ and in complete analogy to the continuous case eqn (24) gives the following form for A :

$$A = \cos\frac{\phi_n}{2} - \sin\frac{\phi_n}{2}\exp\left(\mathfrak{i}\sum_{k=0}^{n}\tau_k\right)\mathfrak{k}$$

with $\phi_n = \angle(S_n, S_{n+1})$ the folding angles and τ_n the angles between successive binormals. If we drop the condition that $\mathcal{F}$ should be of unit length we can renormalize A_n to be $1 - \tan\frac{\phi_n}{2}\exp\left(\mathfrak{i}\sum_{k=0}^{n}\tau_k\right)\mathfrak{k} =: 1 - \Psi_n\mathfrak{k}$ with $\Psi_n \in \mathrm{span}(1, \mathfrak{i}) \cong \mathbb{C}$ and $|\Psi_n| = \kappa_n$ the discrete (real) curvature.

Definition 5 *We call* Ψ *the complex curvature*[2] *of the discrete curve* γ.

Discretizations of the Hashimoto flow (4) (i. e. a Hashimoto flow for a discrete arclength parametrized curve) and the isotropic Heisenberg model (eqn (10)) are well known [5] (see also [3] for a good discussion of the topic). In particular a discrete version of (4) is given by:

$$\dot{\gamma}_k = 2\frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1}\rangle} \tag{25}$$

which implies for a discretization of (10)

$$\dot{S}_k = 2\frac{S_{k+1} \times S_k}{1 + \langle S_{k+1}, S_k\rangle} - 2\frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1}\rangle} \tag{26}$$

Let us state the zero curvature representation for this equation too: Equation (26) is the compatibility condition of $\dot{U}_k = V_{k+1}U_k - U_kV_k$ with

$$\begin{aligned} U_k &= \mathbb{I} + \lambda S_k \\ V_k &= -\tfrac{1}{1+\lambda^2}\left(2\lambda^2 \tfrac{S_k + S_{k-1}}{1+\langle S_k, S_{k-1}\rangle} + 2\lambda \tfrac{S_k \times S_{k-1}}{1+\langle S_k, S_{k-1}\rangle}\right) \end{aligned} \tag{27}$$

The solution to the auxiliary problem

$$\begin{aligned} G_{k+1} &= U_k(\lambda)G_k \\ \dot{G}_k &= V_k(\lambda)G_k \end{aligned} \tag{28}$$

can be viewed as the frame to a discrete Hashimoto surface $\gamma_k(t)$ and one has the same Sym formula as in the continuous case:

Theorem 5 *Given a solution* G *to the system (28) the corresponding discrete Hashimoto surface can be obtained up to an euclidian motion by*

$$\gamma_k(t) = (G_k^{-1}\frac{\partial}{\partial\lambda}G_k)|_{\lambda=0}. \tag{29}$$

Proof One has $G_k^{-1}\frac{\partial}{\partial\lambda}G_k|_{\lambda=0} = \sum_{i=0}^{k-1} S_i = \gamma_k$ for fixed time t_0 and

$$(G_k^{-1}\frac{\partial}{\partial\lambda}G_k|_{\lambda=0})_t = (\frac{\partial}{\partial\lambda}V_k(\lambda)|_{\lambda=0}) = 2\frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1}\rangle}.$$

□

To complete the analogy to the smooth case we give a discretization of the NLSE that can be found in [1] (see also [5, 11]):

$$-i\dot{\Psi}_k = \Psi_{k+1} - 2\Psi_k + \Psi_{k-1} + |\Psi_k|^2(\Psi_{k+1} + \Psi_{k-1}). \tag{30}$$

[2] It would be more reasonable to define $A = 1 - \frac{\Psi_n}{2}\mathfrak{k}$. which implies $\kappa_n = 2\tan\frac{\phi_n}{2}$ but notational simplicity makes the given definition more convenient.

Theorem 6 *Let γ be a discrete arclength parametrized curve. If γ evolves with the discrete Hashimoto flow (25) then its complex curvature Ψ evolves with the discrete nonlinear Schrödinger equation (30)*

A proof of this theorem can be found in [8] and [7]. There is another famous discretization of the NLSE in literature that is related to the dIHM [9, 5]. Again in [7] it is shown that it is in fact gauge equivalent to the above cited which turns out to be more natural from a geometric point of view.

3.1 Discrete elastic curves

As mentioned in Section 2.1 the stationary solutions of the NLSE (i. e. the curves that evolve by rigid motion under the Hashimoto flow) are known to be the *elastic curves.* They have a natural discretization using this property:

Definition 6 *A discrete elastic curve is a curve γ for which the evolution of γ_n under the Hashimoto flow (25) is a rigid motion which means that its tangents evolve under the discrete isotropic Heisenberg model (26) by rigid rotation.*

In [3] Bobenko and Suris showed the equivalence of this definition to a variational description.

The fact that (26) has to be a rigid rotation means that the left hand side must be $S_n \times p$ with a unit imaginary quaternion p. We will now give a description of elastic curves by their complex curvature function only:

Theorem 7 *The complex curvature Ψ_n of a discrete elastic curve γ_n satisfies the following difference equation:*

$$\mathcal{C}\frac{\Psi_n}{1+|\Psi_n|^2} = \Psi_{n+1} + \Psi_{n-1} \tag{31}$$

for some real constant $\mathcal{C}$.

Equation (31) is a special case of a discrete-time Garnier system (see [10]).

Proof One can proof the theorem by direct calculations or using the equivalence of the dIHM model and the dNLSE stated in theorem 6. If the curve γ evolves by rigid motion its complex curvature may vary by a phase factor only: $\Psi(x,t) = e^{i\lambda(t)}\Psi(x,t_0)$ or $\dot{\Psi} = i\dot{\lambda}\Psi$. Plugging this in eqn (30) gives

$$-\dot{\lambda}\Psi_k = \Psi_{k+1} - 2\Psi_k + \Psi_{k-1} + |\Psi_k|^2(\Psi_{k+1} + \Psi_{k-1})$$

which is equivalent to (31) with $\mathcal{C} = 2 - \dot{\lambda}$.

□

As an example Fig 2 shows two discretizations of the elastic figure eight.

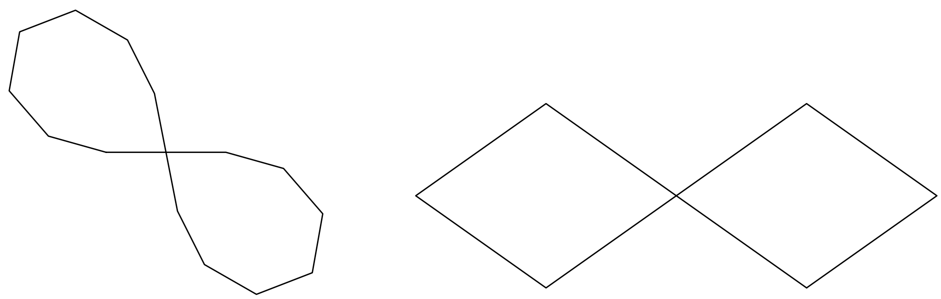

Figure 2: Two discretizations of the elastic figure eight.

3.2 Bäcklund transformations for discrete space curves and Hashimoto surfaces

3.2.1 Algebraic description

In complete analogy to Section 2.2.1 we state

Theorem 8 *Let G_k be a solution to equations (28) with U_k and V_k as in (27) (i. e. $U(1)-\mathbb{I}$ solves the dIHM model). Choose $\lambda_0, s_0 \in \mathbb{C}$. Then $\widetilde{G}_k(\lambda) := B_k(\lambda)G_k(\lambda)$ with $B_k(\lambda) = (\mathbb{I}+\lambda\rho_k), \rho_k \in \mathbb{H}$ defined by the conditions that $\lambda_0, \overline{\lambda}_0$ are the zeroes of* $\det(B_k(\lambda))$ *and*

$$\widetilde{G}_k(\lambda_0)\begin{pmatrix} s_0 \\ 1 \end{pmatrix} = 0 \quad \text{and} \quad \widetilde{G}_k(\overline{\lambda}_0)\begin{pmatrix} 1 \\ -\overline{s}_0 \end{pmatrix} = 0 \tag{32}$$

solves a system of the same type. In particular

$$\widetilde{U}_k(1) - \mathbb{I} = \widetilde{G}_x(1)\widetilde{G}^{-1}(1) - \mathbb{I}$$

solves again the discrete Heisenberg magnet model (26).

Proof Analogous to the smooth case. □

Example Let us dress the (this time discrete) straight line again: We set $S_n \equiv \mathfrak{i}$ and get

$$\begin{aligned} G_n(\lambda) &= (\mathbb{I}+\lambda\mathfrak{i})^n \exp(-2\tfrac{\lambda^2}{1+\lambda^2}t\mathfrak{i}) \\ &= \begin{pmatrix} (1+i\lambda)^n e^{-2i\frac{\lambda^2}{1+\lambda^2}t} & 0 \\ 0 & (1-i\lambda)^n e^{2i\frac{\lambda^2}{1+\lambda^2}t} \end{pmatrix}. \end{aligned}$$

After choosing λ_0 and s_0 and writing again $\rho = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ we get with the shorthands $p = (1+i\lambda_0)^n e^{-2i\frac{\lambda_0{}^2}{1+\lambda_0{}^2}t}$ and $q = (1-i\lambda_0)^n e^{2i\frac{\lambda_0{}^2}{1+\lambda_0{}^2}t}$

$$\begin{aligned} p &= -\lambda_0(pa + s_0qb) \\ q &= \lambda_0(p\overline{b} - s_0q\overline{a}) \end{aligned}$$

which can be solved for a and b :

$$\begin{aligned} a &= -\frac{\frac{1}{\lambda_0}\frac{\overline{p}}{\overline{q}}+\frac{s_0\overline{s}_0}{\overline{\lambda}_0}\frac{q}{p}}{\frac{\overline{p}}{\overline{q}}+s_0\overline{s}_0\frac{q}{p}} \\ b &= \overline{s}_0\frac{\frac{1}{\overline{\lambda}_0}-\frac{1}{\lambda_0}}{\frac{\overline{p}}{\overline{q}}+s_0\overline{s}_0\frac{q}{p}}. \end{aligned} \tag{33}$$

Again we can write the formula for the curve $\widetilde{\gamma}$:

$$\widetilde{\gamma}_n = \operatorname{Im}(\rho_n) + \gamma_n = \begin{pmatrix} \operatorname{Im}(a_n) + in & b_n \\ -\overline{b}_n & -\operatorname{Im}(a_n) - in \end{pmatrix}.$$

Figure 3 shows two solutions with $s_0 = 0.5 + i$ and $\lambda_0 = 0.4 - 0.4i$ and $\lambda_0 = -0.4i$

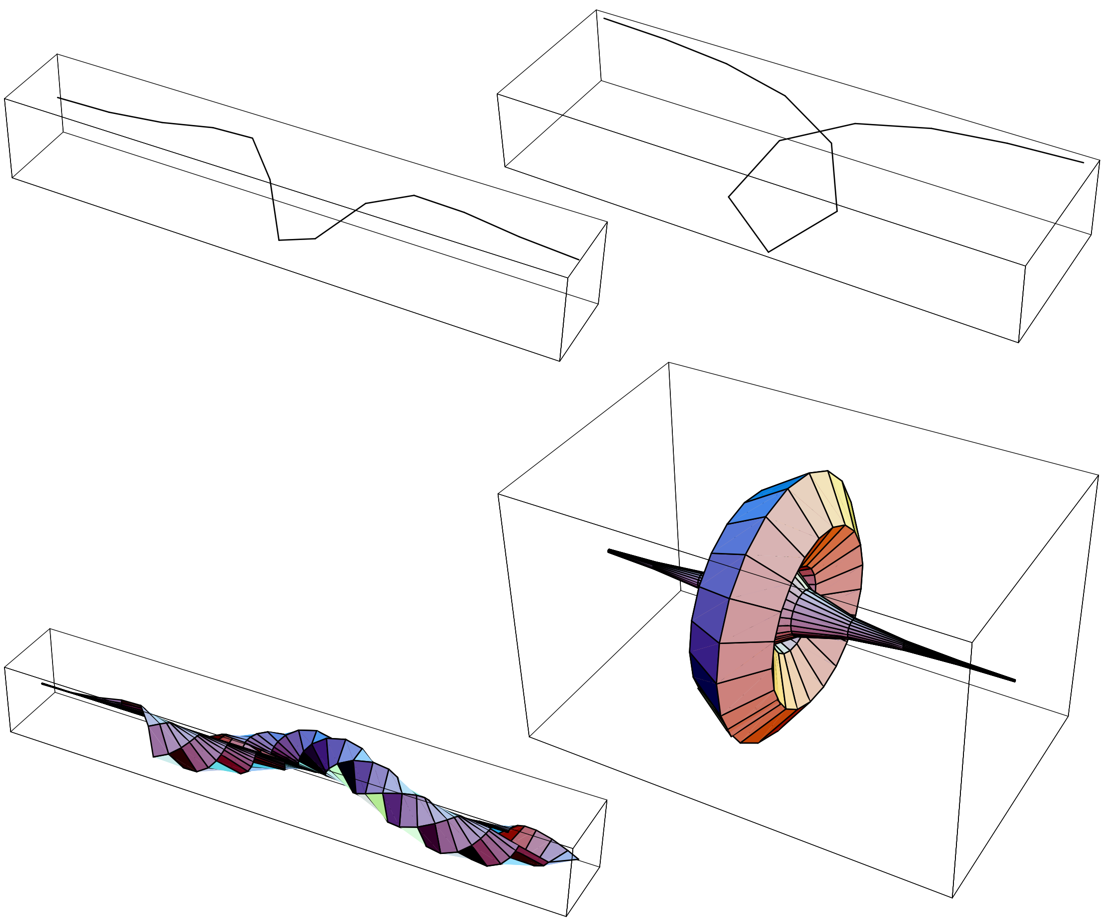

Figure 3: Two discrete dressed straight lines and the corresponding Hashimoto surfaces

respectively. The second one is again planar. Note the strong similarity to the smooth examples in Figure 1.

Of course one has again a permutability theorem:

Theorem 9 (Bianchi permutability) *Let $\widetilde{\gamma}$ and $\widehat{\gamma}$ be two Bäcklund transforms of γ. Then there is a unique discrete Hashimoto surface $\widehat{\widetilde{\gamma}}$ that is Bäcklund transform of $\widetilde{\gamma}$ and $\widehat{\gamma}$.*

Proof Literally the same as for theorem 3. $\square$

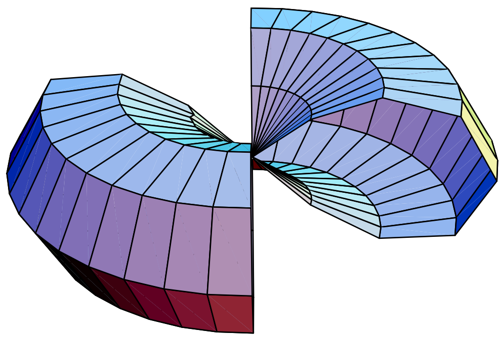

Figure 4: The Hashimoto surface from a discrete elastic eight.

3.2.2 Geometry of the discrete Bäcklund transformation

In this section we want to derive the discrete Bäcklund transformations by mimicing the twisted Traktrix construction from Lemma 4:

Let $\gamma : \mathbb{Z} \to \mathrm{Im}\,\mathbb{H}$ be an discrete arclength parametrized curve. To any initial vector v_n of length l there is a S^1-family of vectors v_{n+1} of length l satisfying $|\gamma_n + v_n - (\gamma_{n+1} + v_{n+1})| = 1$. This is basically folding the parallelogram spanned by v_n and S_n along the diagonal $S_n - v_n$. To single out one of these new vectors let us fix the angle δ_1 between the planes spanned by v_n and S_n and v_{n+1} and S_n (see Fig. 5). This furnishes a unique evolution of an initial v_0 along γ. The polygon $\widetilde{\gamma}_n = \gamma_n + v_n$ is again a discrete arclength parametrized curve which we will call a *Bäcklund transform* of γ.

There are two cases in which the elementary quadrilaterals $(\gamma_n, \gamma_{n+1}, \widetilde{\gamma}_{n+1}, \widetilde{\gamma}_n)$ are planar. One is the parallelogram case. The other can be viewed as a discrete version of the Traktrix construction.

Definition 7 *Let γ be a discrete arclength parametrized curve. Given δ_1 and v_0, $|v_0| = l$ there is a unique discrete arclength parametrized curve $\widetilde{\gamma}_n = \gamma_n + v_n$ with $|v_n| = l$ and $\angle(\mathrm{span}(v_n, S_n), \mathrm{span}(v_{n+1}, S_n)) = \delta_1$.*

$\widetilde{\gamma}$ *is called a* Bäcklund transform *of γ and $\widehat{\gamma} = \gamma + \frac{1}{2}v$ is called a* discrete twisted Traktrix. *for γ (and $\widetilde{\gamma}$).*

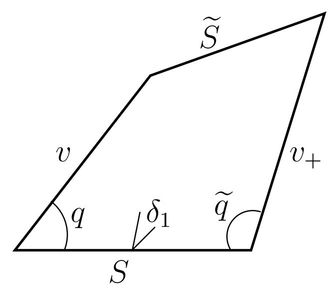

Figure 5: An elementary quadrilateral of the discrete Bäcklund transformation

Remark Note that in case of $\delta = \pi$ the $\mathrm{cr}(\gamma, \widetilde{\gamma}, \widetilde{\gamma}_+, \gamma_+) = l^2$.
Of course we will show, that this notion of Bäcklund transformation coincides with the one from the last section. Let us investigate this Bäcklund transformation in greater detail. For now we do not restrict our selves to arclength parametrized curves. We state the following

Lemma 10 *The map M sending v_n to v_{n+1} in above Bäcklund transformation is a Möbius transformation.*

Proof Let us look at an elementary quadrilateral: For notational simplicity let us write $S = \gamma_{n+1} - \gamma_n$, $\widetilde{S} = \widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$, $|S| = s$, $v = v_n$, and $v_+ = v_{n+1}$. If we denote the angles $\angle(S, v)$ and $\angle(v_+, S)$ with q and $\widetilde{q}$, we get

$$e^{i\widetilde{q}} = \frac{ke^{iq} - 1}{e^{iq} - k} \tag{34}$$

with $k = \tan\frac{\delta_1}{2}\tan\frac{\delta_2}{2}$ and δ_1 as in Fig. 5. δ_2 is the corresponding angle along the edge v. Note that l, s, k, δ_1, and δ_2 are coupled by

$$k = \tan\frac{\delta_1}{2}\tan\frac{\delta_2}{2} \qquad \frac{l}{s} = \frac{\sin\delta_2}{\sin\delta_1}. \tag{35}$$

To get an equation for v_+ from this we need to have all vectors in one plane. So set $\sigma = \cos\frac{\delta_1}{2} + \sin\frac{\delta_1}{2}\frac{S}{s}$. Then conjugation with σ is a rotation around S with angle δ_1. If we replace e^{iq} by $\frac{\sigma v \sigma^{-1}}{l}\left(\frac{S}{s}\right)^{-1}$ and $e^{i\widetilde{q}}$ by $-\frac{S}{s}v_+^{-1}l$ equation (34) becomes quaternionic but stays valid (one can think of it as a complex equation with different "i"). Equation (34) now reads

$$\frac{v_+S}{ls} = \frac{\frac{s}{l}\sigma v\sigma^{-1}S^{-1} - k}{\frac{ks}{l}\sigma v\sigma^{-1}S^{-1} - 1}.$$

We can write this in homogenous coordinates: $\mathbb{H}^2$ carries a natural right $\mathbb{H}$-modul structure, so one can identify a point in $\mathbb{HP}^1$ with a quaternionic line in $\mathbb{H}^2$ by $p \cong (r, s) \iff p = rs^{-1}$. In this picture our equation gets

$$\begin{pmatrix} \frac{1}{ls}v_+S \\ 1 \end{pmatrix}\lambda = \begin{pmatrix} \frac{s}{l}\sigma & -kS\sigma \\ \frac{ks}{l}\sigma & -S\sigma \end{pmatrix}\begin{pmatrix} v \\ 1 \end{pmatrix}.$$

Bringing ls and S on the right hand side gives us finally the matrix

$$\mathcal{M} := \begin{pmatrix} \frac{1}{k}\sigma & -\frac{l}{s}S\sigma \\ \frac{1}{ls}S\sigma & \frac{1}{k}\sigma \end{pmatrix}. \tag{36}$$

Since we know that this map sends a sphere of radius l onto itself, we can project this sphere stereographically to get a complex matrix. The matrix

$$P = \begin{pmatrix} 2\mathfrak{i} & -2l \\ \frac{\mathfrak{j}}{l} & \mathfrak{k} \end{pmatrix}$$

projects lS^2 onto $\mathbb{C}$. Its inverse is given by

$$P^{-1} = -\frac{1}{4}\begin{pmatrix} \mathfrak{i} & 2l\mathfrak{j} \\ \frac{1}{l} & 2\mathfrak{k} \end{pmatrix}.$$

One easily computes

$$\mathcal{M}_{\mathbb{C}} = P\mathcal{M}P^{-1} = -\frac{1}{4}\begin{pmatrix} \nu + i\,\mathrm{Re}(S\mathfrak{i}) & 2l\,\mathrm{Im}(S\mathfrak{i})\mathfrak{j} \\ -\frac{1}{2l}\overline{\mathrm{Im}(S\mathfrak{i})\mathfrak{j}} & \nu - i\,\mathrm{Re}(S\mathfrak{i}) \end{pmatrix} \tag{37}$$

with $\nu = is\frac{\tan\frac{\delta_1}{2}i - \frac{1}{k}}{\frac{\tan\frac{\delta_1}{2}}{k}i - 1}$. This completes our proof. ☐

Remark

- Using equation (35) one can compute

$$\nu = s\tan\frac{\delta_1}{2}\frac{1-k^2}{\tan^2\frac{\delta_1}{2}+k^2} + il = l\tan\frac{\delta_2}{2}\frac{1-k^2}{\tan^2\frac{\delta_2}{2}+k^2} + il. \tag{38}$$

  So the real part of ν is invariant under the change $s \leftrightarrow l$, $\delta_1 \leftrightarrow \delta_2$. Therefore instead of thinking of $\widetilde{S}$ as an transform of S with parameter ν one could view v_+ a transform of v with parameter $\nu + i(s-l)$.

- One can gauge $\mathcal{M}_{\mathbb{C}}$ to get rid of the off-diagonal $2l$ factors

$$M = \begin{pmatrix} \frac{1}{\sqrt{2l}} & 0 \\ 0 & \sqrt{2l} \end{pmatrix}\mathcal{M}_{\mathbb{C}}\begin{pmatrix} \sqrt{2l} & 0 \\ 0 & \frac{1}{\sqrt{2l}} \end{pmatrix}.$$

  Then we can write in abuse of notation

$$M = \nu\mathbb{I} - S \tag{39}$$

  Here $\nu\mathbb{I}$ is no quaternion if ν is complex. The eigenvalues of $\mathcal{M}_{\mathbb{C}}$ and M clearly coincide and M obviously coincides with the Lax matrix U_k of the dIHM model in equation (27) up to a factor $\frac{1}{\nu}$ with $\lambda = -\frac{1}{\nu}$.

As prommised the next lemma shows that the geometric Bäcklund transformation discussed in this section coincides with the one from the algebraic description.

Lemma 11 *Let $S, v \in \operatorname{Im}\mathbb{H}$ be nonzero vectors , $|v| = l$, $\widetilde{S}$ and v_+ be the evolved vectors in the sense of our Bäcklund transformation with parameter ν ($\operatorname{Im}\nu = l$). then*

$$(\lambda\mathbb{I} + \widetilde{S})(\lambda\mathbb{I} + \operatorname{Re}\nu + v) = (\lambda\mathbb{I} + \operatorname{Re}\nu + v_+)(\lambda\mathbb{I} + S) \tag{40}$$

holds for all λ.

Proof Comparing the orders in λ on both sides in equation (40) gives two equations

$$\widetilde{S} + \operatorname{Re}\nu + v = \operatorname{Re}\nu + v_+ + S \tag{41}$$

$$\widetilde{S}(\operatorname{Re}\nu + v) = (\operatorname{Re}\nu + v_+)S. \tag{42}$$

The first holds trivially from construction the second gives

$$\operatorname{Re}\nu = (v_+S - \widetilde{S}v)(\widetilde{S} - S)^{-1}.$$

This can be checked by elementary calculations using equation (38) for the real part of ν. □

Like in the continuous case we can deduce that $\operatorname{Im}(\rho_n) = v_n = \widetilde{\gamma}_n - \gamma_n$ which gives us the constant distance between the original curve γ_n and its Bäcklund transform $\widetilde{\gamma}_n$.

4 The doubly discrete Hashimoto flow

From now on let $\gamma : \mathbb{Z} \to \operatorname{Im}\mathbb{H}$ be periodic or have at least periodic tangents $S_n = \gamma_{n+1} - \gamma_n$ with period N (we will see later that rapidly decreasing boundary conditions are valid also). As before let $\widetilde{\gamma}$ be a Bäcklund transform of γ with initial point $\widetilde{\gamma}_0 = \gamma_0 + v_0$, $|v_0| = l$. As we have seen the map sending v_n to v_{n+1} is a Möbius transformation and therefore the map sending v_0 to v_N is one too. As such it has in general two but at least one fix point. Thus starting with one of them as initial point the Bäcklund transform $\widetilde{\gamma}$ is periodic too (or has periodic tangents S). Clearly this can be iterated to get a discrete evolution of our discrete curve γ.

Lemma 12 *Let γ be a discrete curve with periodic tangents S of period N. Then the tangents $\widetilde{S}$ of a dressed curve $\widetilde{\gamma}$ with the parameters λ_0 and s_0 are again periodic if and only if the vector $(1, s_0)$ is an eigenvector of the monodromy matrix $G_N(\lambda)$ at $\lambda = \lambda_0$.*

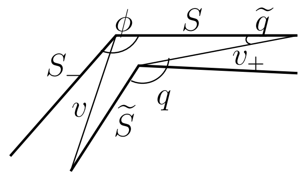

Figure 6: An elementary quadrilateral if $l = 1$ and $\delta_1 \approx \frac{\pi}{2}$

Proof We use the notation from Theorem 8. Since $\widetilde{\gamma}_n - \gamma_n = v_n = \mathrm{Im}(\rho_n)$ and since $B(\lambda) = \mathbb{I} + \lambda\rho$ is completely determined by λ_0 and v we have, that $B_0(\lambda) = B_N(\lambda)$. On the other hand on can determine $B(\lambda)$ by λ_0 and s_0. Since $G_0(\lambda) = \mathbb{I}$ condition 32 says that $\binom{1}{s_0}$ and $G_n(\lambda_0)\binom{1}{s_0}$ must lie in $\ker B_0(\lambda_0)$. $\square$

A Lax representation for this evolution is given by equation (42) which is basically the Bianchi permutability of the Bäcklund transformation.

In the following we will show that for the special choice $l = 1$ and $\delta_1 \approx \frac{\pi}{2}$ the resulting evolution can be viewed as a discrete smoke ring flow. More precisely one has to apply the transformation twice: once with δ_1 and once with $-\delta_1$. In [7] it is shown, that under this evolution the complex curvature of the discrete curve solves the doubly discrete NLSE introduced by Ablowitz and Ladik [2], which of course is an other good argument.

Proposition 13 *A Möbius transformation that sends a disc into its inner has a fix point in it.*

Proof For the Möbius transformation M look at the vector field f given by $f(x) = M9x) - x$. This must have a zero. $\square$

Now we show the following

Lemma 14 *If $\angle(-S_-, v) \leq \epsilon$, ϵ sufficiently small, there exists a δ_1 such that $\angle(-S, v_+) < \epsilon$.*

Proof With notations as in Fig 6 we know $e^{i\widetilde{q}} = \frac{ke^{iq}-1}{k-e^{iq}}$ and $q \in [\phi - \epsilon, \phi + \epsilon]$ giving us

$$2i \sin \widetilde{q} = 2i \,\mathrm{Im}\, e^{i\widetilde{q}} = 2i \frac{(k^2 - 1)\sin(\phi \pm \epsilon)}{(k^2 + 1) - 2k\cos(\phi \pm \epsilon)}$$

which proofs the claim since k goes to 1 if δ_1 tends to $\frac{\pi}{2}$. $\square$

Knowing this one can see that an initial v_0 with $\angle(-S_{N-1}, v_0) \leq \epsilon$ is mapped to a v_N with $\angle(-S_{N-1}, v_N) < \epsilon$. Above Proposition gives that there must be a fix point p_0 with $\angle(-S_{N-1}, p_0) < \epsilon$.

But if $p_n \approx -S_{n-1}$ we get $\widetilde{\gamma}_n \approx \gamma_{n-1}$ and $\widetilde{\gamma}_n - \gamma_{n-1}$ is close to be orthogonal to $\mathrm{span}(S_{n-2}, S_{n-1})$. So it is a discrete version of an evolution in binormal direction —plus a shift. To get rid of this shift, one has to do the transformation twice but with opposite sign for δ_1. Figure 7 shows some stages of the smooth Hashimoto

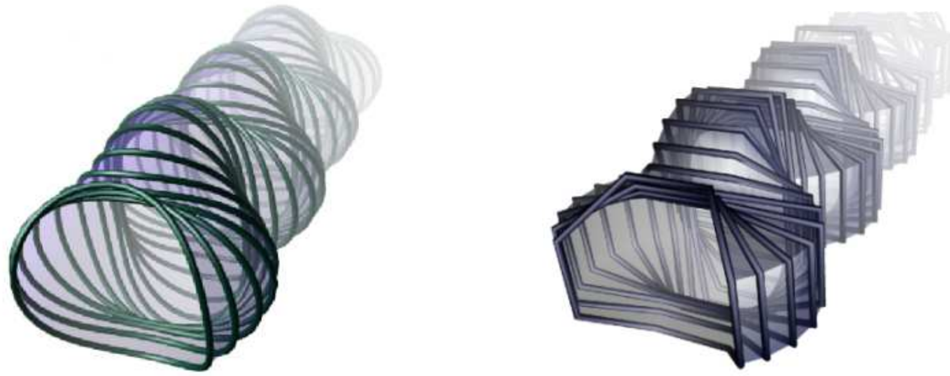

Figure 7: An oval curve under the Hashimoto flow and the discrete evolution of its discrete pendant.

flow for an oval curve and the discrete evolution of its discrete counterpart. In general the double Bäcklund transformation can be viewed as a discrete version of a linear combination of Hashimoto and tangential flow—this is emphasized by the fact that the curves that evolve under such a linear combination by rigid motion only coincide in the smooth and discrete case:

4.1 discrete Elastic Curves

As a spin off of the last section one can easily show, that the elastic curves defined in Section 3 as curves that evolve under the Hashimoto flow by rigid motion only do the same for the doubly discrete Hashimoto flow. Again we will use the evolution of the complex curvature of the discrete curve. We mentioned before that in the doubly discrete case the complex curvature evolves with the doubly discrete NLSE given by Ablowitz and Ladik [2, 7].

We start by quoting a special case of their result which can be summarized in the following form (see also [11])

Theorem 15 (Ablowitz and Ladik) *given*

$$L_n(\mu) = \begin{pmatrix} \mu & q_n \\ -\overline{q}_n & \mu^{-1} \end{pmatrix}$$

and $V_n(\mu)$ with the following μ–dependency:

$$V_n(\mu) = \mu^{-2}V_{-2_n} + \mu^{-1}V_{-1_n} + V_{0n} + \mu^1 V_{1n} + \mu^2 V_{2n}.$$

Then the zero curvature condition $V_{n+1}(\mu)L_n(\mu) = \widetilde{L}_n(\mu)V_n(\mu)$ gives the following equations:

$$\begin{aligned}
(\widetilde{q}_n - q_n)/i &= \alpha_+ q_{n+1} - \alpha_0 q_n + \overline{\alpha}_0 \widetilde{q}_n - \overline{\alpha}_+ \widetilde{q}_{n-1} \\
&\quad +(\alpha_+ q_n \mathcal{A}_{n+1} - \overline{\alpha}_+ \widetilde{q}_n \overline{\mathcal{A}}_n) \\
&\quad +(-\overline{\alpha}_- \widetilde{q}_{n+1} + \alpha_- q_{n-1})(1+|\widetilde{q}_n|^2)\Lambda_n \\
\mathcal{A}_{n+1} - \mathcal{A}_n &= \widetilde{q}_n \overline{\widetilde{q}}_{n-1} - q_{n+1}\overline{q}_n \\
\Lambda_{n+1}(1+|q_n|^2) &= \Lambda_n(1+|\widetilde{q}_n|^2)
\end{aligned} \tag{43}$$

with constants α_+, α_0 and α_-.

In the case of periodic or rapidly decreasing boundary conditions the natural conditions $\mathcal{A}_n \to 0$, and $\Lambda_n \to 1$ for $n \to \pm\infty$ give formulas for $\mathcal{A}_n$ and Λ_n:

$$\mathcal{A}_n = q_n \overline{q}_{n-1} + \sum_{j=j_0}^{n-1} (q_j \overline{q}_{j-1} - \widetilde{q}_j \overline{\widetilde{q}}_{j-1})$$

$$\Lambda_n = \prod_{j=j_0}^{n-1} \frac{1+|\widetilde{q}_j|^2}{1+|q_j|^2}$$

with $j_0 = 0$ in the periodic case and $j_0 = -\infty$ in case of rapidly decreasing boundary conditions.

Theorem 16 *The discrete elastic curves evolve by rigid motion under the doubly discrete Hashimoto flow.*

Proof Evolving by rigid motion means for the complex curvature of a discrete curve, that it must stay constant up to a possible global phase, i. e. $\widetilde{\psi}_n = e^{2i\theta}\Psi_n$. Due to Theorem 15 the evolution equation for ψ_n reads

$$\begin{aligned}
\tfrac{(\widetilde{\Psi}_n - \Psi_n)}{i} &= \alpha_+ \Psi_{n+1} - \alpha_0 \Psi_n + \overline{\alpha}_0 \widetilde{\Psi}_n - \overline{\alpha}_+ \widetilde{\Psi}_{n-1} + (\alpha_+ \Psi_n \mathcal{A}_{n+1} \\
&\quad - \overline{\alpha}_+ \widetilde{\Psi}_n \overline{\mathcal{A}}_n) + (-\overline{\alpha}_- \widetilde{\Psi}_{n+1} + \alpha_- \Psi_{n-1})(1+|\widetilde{\Psi}_n|^2)\Lambda_n
\end{aligned}$$

Using $e^{-i\theta}\widetilde{\psi}_n = e^{i\theta}\Psi_n$ gives $\Delta_n = 1$, $\mathcal{A}_n = e^{2i\theta}\Psi_n \overline{\Psi}_{n-1}$, and finally

$$2\left(\sin\theta + \mathrm{Re}(e^{i\theta}\alpha_0)\right) \frac{\Psi_n}{1+|\Psi_n|^2} =$$

$$= \left(e^{i\theta}\alpha_+ + \overline{e^{i\theta}\alpha_-}\right)\Psi_{n+1} + \left(\overline{e^{i\theta}\alpha_+} + e^{i\theta}\alpha_-\right)\Psi_{n-1}.$$

So the complex curvature of curves that move by rigid motion solve

$$\mathcal{C}\frac{\Psi_n}{1+|\Psi_n|^2} = e^{i\mu}\Psi_{n+1} + e^{-i\mu}\Psi_{n-1} \tag{44}$$

with some real parameters $\mathcal{C}$ and μ which clearly holds for discrete elastic curves. $\square$

Remark The additional parameter μ in eqn (44) is due to the fact that the Ablowitz Ladik system is the general double Bäcklund transformation and not only the one with parameters ν and $-\overline{\nu}$. This is compensated by the extra torsion μ and the resulting curve is in the associated family of an elastic curve. These curves are called *elastic rods* [3].

4.2 Bäcklund transformations for the doubly discrete Hashimoto surfaces

Since the doubly discrete Hashimoto surfaces are build from Bäcklund transformations themselves the Bianchi permutability theorem (Theorem 9) ensures that the Bäcklund transformations for discrete curves give rise to Bäcklund transformations for the doubly discrete Hashimoto surfaces too. Thus every thing said in section 3.2 holds in the doubly discrete case too.

Conclusion

We presented an integrable doubly discrete Hashimoto or Heisenberg flow, that arises from the Bäcklund transformation of the (singlely) discrete flow and showed how the equivalence of the discrete and doubly discrete Heisenberg magnet model with the discrete and doubly discrete nonlinear Schrödinger equation can be understood from the geometric point of view. The fact that the stationary solutions of the dNLSE and the ddNLSE coincide stresses the strong similarity of the both and the power of the concept of integrable discrete geometry.

Let us end by giving some more figures of examples of the doubly discrete Hashimoto flow.

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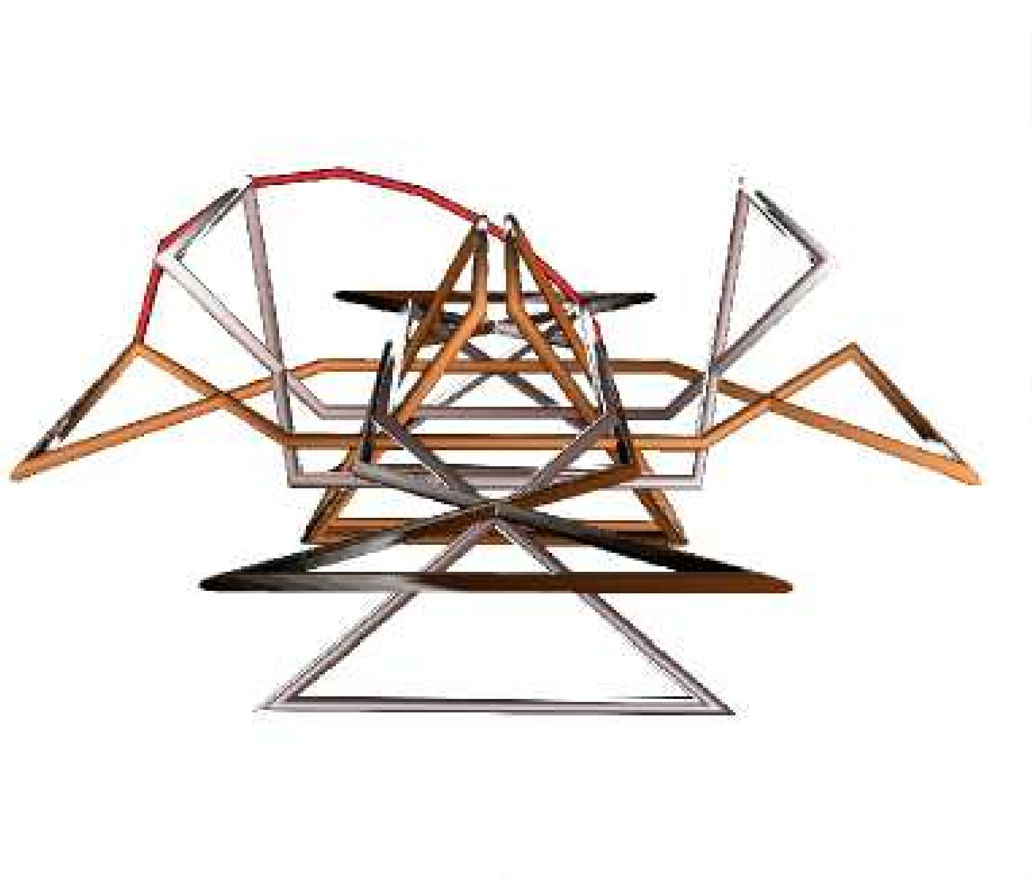

Figure 8: A discrete double eight that gives a Hashimoto torus. The red line is the trace of one vertex.

Figure 9: The doubly discrete Hashimoto flow on a equal sided triangle with subdivided edges.

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